Just *perfect*

part 1

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Introduction

This article is about a very small subset of the positive integers. We say that the positive integer N is *perfect* if it is the sum of all its divisors, including 1, but less that N itself. (Notice that N is technically a divisor of itself.)

For example, $N = 6$ is perfect, because the (relevant) divisors are 1, 2 and 3, and $6 = 1 + 2 + 3$. On the other hand, $N = 12$ has divisors 1, 2, 3, 4 and 6, but since $1 + 2 + 3 + 4 + 6 = 16 \neq 12$, 12 is not a perfect number.

Table 1

N	Divisors	s(N)
2	1	1
3	1	1
4	1, 2	3
5	1	1
6	**1, 2, 3**	**6**
7	1	1
8	1, 2, 4	7
9	1, 3	4
…	…	…

Perfect numbers are not new: in fact the search for them began in ancient times. The first three perfect numbers were known to the ancient mathematicians at least from the time of Pythagoras (circa 500 BC).

Now if we are going to study the properties of the perfect numbers, we need more than one — and the ancient Greeks have laid down the challenge of finding at least three.

Finding perfect numbers

The most obvious first plan is to set up a table as shown in Table 1, with columns for the numbers N. lists of the divisors of N, and the divisor sums $s(N)$.

It is easy to fill in the entries, but you will need a little perseverance to find the next perfect number. What is it?

To find the third perfect number, my hint is: try a different method! If you have any computing experience, it is easy and satisfying to construct a little program for generating the first few perfect numbers. The program given in Figure 1 is in Pascal, but the notes alongside indicate the purpose of each line of code.

This program can be improved. For example, we do not need to test for $d = 1$ up to $N - 1$; up to $N/2$ will do, but of course $N/2$ may not be an integer.

```
program perfect
{to list the perfect numbers up to 10000}

  var  s, d, N: integer;

  begin {program}
    for N := 2 to 10000 do
      begin{for}
      s:=0;
      for d:= 1 to (N-1) do
        begin{for}
        if N mod d = 0 then  S:= S + d;
      end{for};
      if s = N then writeln(N)
    end{for};
  end. {program}
```

Set variables for the number *N*, an arbitrary divisor *d*, and divisor sum *s*.

Set the range of testing for numbers *N*. For each *N* we will build the sum *s*, starting with $s = 0$. To do this, we divide *N* successively by $d = 1, 2, 3, \ldots$ If *d* divides *N* exactly, we add it to *s*.

If the *s* we obtain equals *N* we write it down. Now go back to the beginning for the next *N*.

Figure 1

The program in Figure 1 generates the short list:

6
28
496
8128

— the first four perfect numbers. You might like to check that

$$28 = 1 + 2 + 4 + 7 + 14.$$

In fact, the fifth and sixth perfect numbers are

33 550 336
8 589 869 056

This is still a fairly small sample, but it is enough for us to make some conjectures (intelligent guesses).

Conjectures and questions

1. Look at the six numbers. What can you say about the parity of the numbers? about the last digits? Are there any odd perfect numbers?

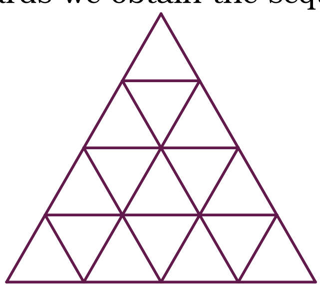
2. We have seen that $6 = 1 + 2 + 3$. Do the other numbers have a similar representation? In fact, there are many numbers of this form. Can we tell from the sum which of these are perfect?

3. Let's try factoring the early numbers.

$$6 = 2 \times 3$$
$$28 = 4 \times 7$$
$$496 = 16 \times 31$$
$$8128 = \ldots$$

Is there a pattern here? What is it?

4. Triangular numbers are numbers created by counting the vertices in a triangular grid. So in the diagram below, starting from the bottom left hand corner and working right and upwards we obtain the sequence:



1, 3, 6, 10, 15, 21, 28, ...

What do you notice? Do you expect other perfect numbers to appear? Why?

5. If we start with the second of the perfect numbers we have:

$$28 = 1^3 + 3^3$$
$$496 = 1^3 + 3^3 + 5^3 + 7^3$$

Is there an ongoing pattern here?

6. Perfect numbers greater than 6 also show other curious patterns. Let us try adding together the digits, then adding together the digits of this sum, and so on. For example:

28: $2 + 8 = 10$
$1 + 0 = 1$

496: $4 + 9 + 6 = 19$
$1 + 9 = 10$
$1 + 0 = 1$

8128: $8 + 1 + 2 + 8 = 19$
$1 + 9 = 10$
$1 + 0 = 1$

In these examples, we always finish up with a 1. Does this always happen?

7. Suppose for any perfect number N we take the reciprocals of the divisors of N which are less than N. We obtain:

6: $\frac{1}{2} + \frac{1}{3} + \frac{1}{6} = 1$
28: $\frac{1}{2} + \frac{1}{4} + \frac{1}{7} + \frac{1}{14} + \frac{1}{28} = 1$
496: $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{31} + \frac{1}{62} + \frac{1}{124} + \frac{1}{248} + \frac{1}{496} = 1$

Do the reciprocals of a perfect number always add to 1? In fact this is rather a trivial result; you should be able to show quite easily that the answer to this question is "Yes".

Clearly the perfect numbers are a remarkable resource for making conjectures. Some of the questions we have asked remain unresolved. Others we can answer, and we shall return to this in the next issue.

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From Helen Prochazka's Scrapbook

Mathematics, rightly viewed, possesses a beauty cold and austere like that of a sculpture, without any appeal to any part of our weaker nature, without the gorgeous trappings of painting or music, yet sublimely pure and capable of a stern perfection such as only the greatest art can show.
Bertrand Russel, 20th century mathematician

May not music be described as the mathematics of sense, mathematics as music of the reason? The musician feels mathematics, the mathematician thinks music — music the dream, mathematics the working life.
Joseph Sylvester, 19th century mathematician

When mathematics is taught, it is presented mainly as a collection of slightly related techniques and manipulations. The profound, yet simple concepts get little attention. If art was taught in the same way, it would consist mostly of learning how to chip stone and mix paints.
George Boehm, 20th century educator

Just *perfect* part 2

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The numbers

In the last issue, we defined a perfect number N to be one for which the sum of the divisors d $(1 \leq d < N)$ is N. We gave the first few perfect numbers, starting with those known by the early Greeks. Here we give an extended list, with some comments about their discovery.

The early Greeks were aware of the first four perfect numbers:

$P_1 = 6$
$P_2 = 28$
$P_3 = 496$
$P_4 = 8128$

Johann Regiomontanus

The Arab mathematician Ibn Fallus (1194–1239) wrote a treatise in which he gave the first seven perfect numbers. So:

$P_5 = 33\ 550\ 336$
$P_6 = 8\ 589\ 869\ 056$
$P_7 = 137\ 438\ 691\ 328$

As the work of Ibn Fallus was not widely known in Europe, these numbers were rediscovered by Regiomontanus (1461), Scheybl (1555) and Cataldi (1603). To the present, some 40 perfect numbers have been discovered.

Structure

Now we might expect to find the early perfect numbers by trial and error, but how did mathematicians obtain the later ones, which clearly get large very rapidly?

Let us review an exercise we did before. You might have found:

$P_1 = 6 \quad = 2 \times 3 \quad = 2^1 \times (2^2 - 1)$
$P_2 = 28 \quad = 4 \times 7 \quad = 2^2 \times (2^3 - 1)$
$P_3 = 496 \quad = 16 \times 31 \quad = 2^4 \times (2^5 - 1)$
$P_4 = 8128 \quad = 64 \times 127 \quad = 2^6 \times (2^7 - 1).$

If you are brave, you can try P_5 and beyond! There is clearly a pretty pattern appearing here, although there are some mysterious gaps. Let us try evaluating the sequence of bracketed terms on the right, including the missing entries:

$(2^2 - 1) = 3$ •
$(2^3 - 1) = 7$ •
$(2^4 - 1) = 15$
$(2^5 - 1) = 31$ •
$(2^6 - 1) = 63$
$(2^7 - 1) = 127$ •

The entries marked with a "•" correspond to our perfect numbers. Before we continue, what do you notice about these entries (that is not true about the remaining two)? Can you make a conjecture?

It looks as though a number of the form $2^n \times (2^{n+1} - 1)$ where the second term is prime will be a perfect number. (A number p is *prime* when its only divisors are 1 and itself. Thus, 3, 7, 31 and 127 are prime numbers. The number 15 is not prime because it is also divisible by 3 and 5.)

Euclid's proof

Perhaps surprisingly, Euclid was able to establish the above conjecture:

> **Theorem** A number $N = 2^n \times (2^{n+1} - 1)$ in which the second term is prime, is a perfect number.

It is not hard to prove this. Let us write

$$M_n = 2^n - 1,$$

and for the purpose of this proof, we set $S(N)$ to be the sum of all the divisors of N, including N itself. This means that for N to be perfect, we require $S(N) = 2N$. For example,

$$S(6) = 1 + 2 + 3 + 6 = 12.$$

In fact, if q is a prime number, then $S(q) = q + 1$. Also,

$$S(2^n) = 1 + 2 + 2^2 + \ldots + 2^n = 2^{n+1} - 1.$$

Now what about the number $N = 2^n \times q$, where $q > 2$ is prime? Listing all the divisors, we have:

$$\begin{aligned} S(N) &= 1+2+2^2+\ldots+2^n+q+2q+2^2q+\ldots+2^nq \\ &= 2^{n+1} - 1 + q(2^{n+1} - 1) \\ &= (2^{n+1} - 1)(1 + q) \end{aligned}$$

Finally, suppose that $q = 2^{n+1} - 1$ (where n must be such that q is a prime). Then

$$\begin{aligned} S(N) &= S\{2^n \times (2^{n+1} - 1)\} \\ &= (2^{n+1} - 1)(1 + 2^{n+1} - 1) \\ &= 2^{n+1}(2^{n+1} - 1) \\ &= 2N, \end{aligned}$$

and N is perfect as required.

There are now two questions to be asked:

- *Are there other even perfect numbers which are not of this form?*
  In a posthumous paper in 1849, Euler provided the first proof that Euclid's construction gives all possible even perfect numbers.
- *Are there any odd perfect numbers?*
  It is not known if any odd perfect numbers exist, although by 1991 all numbers up to 10^{300} (a very large number!) had been checked without success.

Mersenne primes

It is clear that our search for perfect numbers has now changed direction. The test for an even perfect number now depends on finding whether the number $M_n = 2^n - 1$ is prime or not.

Numbers of the form M_n are called *Mersenne numbers*; if this number is a prime, it is called a *Mersenne prime*. It is easy to see that M_n will only be prime if n itself is a prime number. For if $n = rs$, then $M_n = 2^{rs} - 1$ will have a factor $2^r - 1$. This is of the binomial form
$(x^s - 1) = (x - 1)(x^{s-1} + x^{s-2} + \ldots + 1)$.

Mersenne numbers were named after Marin Mersenne (1588–1648) who was a pioneer in the search for prime numbers.

So which values of prime p generate a Mersenne prime M_p? If you enjoyed writing a little program last time to find the smaller perfect numbers, you might like to write another to find the Mersenne primes. There has been a lot of computing work carried out to determine Mersenne primes. This is because prime numbers are the building blocks of our system of integers, and Mersenne primes give an entry into our understanding of primes in general.

Here is a table with some known results.

Prime p	Mersenne number M_p	M_p prime?	Perfect number $2^{p-1}M_p$
2	3	✓	6
3	7	✓	28
5	31	✓	496
7	127	✓	8128
11	1023	✗	–
13	4095	✗	–

You might like to try extending the table to obtain the next perfect number.

Marin Mersenne

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From Helen Prochazka's

Scrapbook

Robert Peirce writing in Lonely Planet Bhutan

In a mountain village named Laya, I was standing in a schoolhouse and staring at a document entitled Manual for Teachers of Mathematics. What caught my eye was something offered as a "first rule".

Always remember that you are a human being as well as a teacher, that your students are human beings, and that you are here because you have something important to give them that they need.

It is not what one would expect as a first rule for maths teachers anywhere else in the world. But this is Bhutan, and in Bhutan, I am learning, people are not the abstract ciphers they can come to be in a more urban environment. They are human beings who, even in official matters, tend to deal with each other as human beings. "I am not as much concerned about the Gross National Product," the king is supposed to have said, "as I am about the Gross National Happiness."