ON PERFECT AND NEAR-PERFECT NUMBERS

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Abstract. We call n a *near-perfect* number, if it is sum of all its proper divisors, except for one of them ("*redundant divisor*"). It is a special kind of Sierpiński number [1]. We prove an Euclid-like theorem for near-perfect numbers and obtain some other results for them.

1. Introduction

A *perfect number* is a positive integer equals to the sum of its proper positive divisors. Denote $\sigma(n)$ the sum of all positive divisors of n. Then n is a perfect number if and only if $\sigma(n) - n = n$, or

$$\sigma(n) = 2n. \tag{1.1}$$

The smallest perfect number is 6, because 1, 2, and 3 are its proper positive divisors, and $1 + 2 + 3 = 6$. By direct experiment one can obtain some first perfect numbers: 6, 28, 496, 8128,... These experiments, at the first time, were conducted by Euclid. Moreover, he was successful to obtain the following important result.

Theorem 1. *(Euclid) If p is such prime that also $2^p - 1$ is prime, then $n = 2^{p-1}(2^p - 1)$ is perfect number.*

It is interesting that 2000 passed before a new large success in the research of perfect numbers. L. Euler was successful to convert the Euclid's theorem for even perfect numbers.

Theorem 2. *(Euler) Even perfect numbers have the form $n = 2^{p-1}(2^p - 1)$, where p and $2^p - 1$ are primes.*

Recall that primes of the form 2^p-1 are called *Mersenne primes.*Up to now it is not known whether exist infinitely many Mersenne primes. Therefore, it is not known whether exist infinitely many even perfect numbers. Not less difficult problem is: whether exists at least one *odd* perfect number? This question is still open as well.

In connection with study of the perfect numbers, it is natural to split all positive integers into three sets: numbers for which $\sigma(n) < 2n$, perfect numbers and numbers for which $\sigma(n) > 2n$. Numbers of the first set are called *deficient*, while numbers of the third set are called *abundant.*

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In contrast to the perfect numbers, it is known that both sets of deficient numbers and abundant numbers are infinite. Many other facts and problems on deficient, abundant and perfect numbers one can find in [1], Chapter B.

A positive integer is called *pseudoperfect,* or *Sierpiński number* (*cf.* [3]), *if it is the sum of some* of its divisors; e.g., $36 = 1 + 2 + 6 + 9 + 18$. In this paper we study Sierpiński numbers of a special kind. We call n a *near-perfect* number, if it is sum of all its proper divisors, except of *one* of them: $d = d(n)$. The latter divisor we call *redundant.* With help of near-perfect numbers, it is suitable, for a given l, to introduce a notion of $(\mathbf{N} \setminus \{l\})$-perfect number. We call n a $(\mathbf{N}\setminus\{l\})$-perfect number if it is perfect number in case of n is not multiple of l, otherwise, if it is near-perfect number with redundant divisor l.

The first near-perfect numbers are (cf. our sequence A181595 in [4]):

$$12, 18, 20, 24, 40, 56, 88, 104, 196, 224, 234, 368, 464, 650, 992, \ldots \tag{1.2}$$

with the redundant divisors (cf. our sequence A181596 in [4])

$$4, 3, 2, 12, 10, 8, 4, 2, 7, 56, 78, 8, 2, 2, 32, \ldots \tag{1.3}$$

2. Euclid-like theorem for near-perfect numbers

For a given $k \geq 1$, consider set $\mathcal{P}_k$ of primes of the form: $2^t - 2^k - 1$, where $t \geq k + 1$.

Theorem 3. *Number* $n = 2^{t-1}(2^t - 2^k - 1)$, *where* $2^t - 2^k - 1 \in \mathcal{P}_k$, *is near-perfect number with redundant divisor* 2^k.

Proof. Since $k \leq t - 1$, then $d = 2^k$ is a proper divisor of n. Besides, $\sigma(n) = (2^t - 1)(2^t - 2^k)$ and, in view of

$$\sigma(n) - 2n = (2^t - 1)(2^t - 2^k) - 2^t(2^t - 2^k - 1) = 2^k,$$

the theorem follows. ■

In contrast to the case of perfect numbers, there exist even near-perfect numbers which have not form $n = 2^{t-1}p$ with $p \in \mathcal{P}_k$. Indeed, consider near-perfect number $n = 650$ from (1.2). The redundant divisor for it is d(650)=2. Nevertheless, 650 is not expressible in the considered form. But Theorem 3 makes plausible the following conjecture.

Conjecture 1. *For given* k, *there exist infinitely many near-perfect numbers with redundant divisor* 2^k.

Let

$$\mathcal{P} = \bigcup_{k=1}^{\infty} \mathcal{P}_k.$$

The first primes from $\mathcal{P}$ are (cf. our sequence A181741 in [4]):

$$3, 5, 7, 11, 13, 23, 29, 31, 47, 59, 61, 127, 191, 223, 239, ... \tag{2.1}$$

Note that all Mersenne primes are in the sequence. Indeed, if a prime p has form $p = 2^r - 1$, then $p = 2^{r+1} - 2^r - 1$. On the other hand, it is easy to see that, if p is in the sequence (2.1), then the representation $p = 2^t - 2^k - 1$, $k >= 1$, $t >= k + 1$, is unique for it.

According to Theorem 3, near-perfect numbers of the form $n = 2^{t-1}(2^t - 2^k - 1)$, where $2^t - 2^k - 1 \in \mathcal{P}_k$, we call *$\mathcal{P}_k$-near-perfect,* for a given k, and *$\mathcal{P}$-near-perfect*, if k is not fixed.

3. Near-perfect numbers generated by perfect numbers

Let us seek near-perfect numbers in the form: $n = 2^x m$, where m is an even perfect number.

Theorem 4. *Number n of the form $n = 2^x m$, where m is even perfect number, is near-perfect if and only if either $x = 1$ or $x = p$, where p is prime such that 2^{p-1} is the most power of 2 dividing m $(2^{p-1}||m)$.*

Proof. Since $2^{p-1}||m$, then, by Theorem 1, $m = 2^{p-1}(2^p - 1)$, where $2^p - 1$ is prime. Therefore, $n = 2^{p+x-1}(2^p - 1)$. We have

$$\sigma(n) - 2n = (2^{x+p} - 1)2^p - 2^{x+p}(2^p - 1) = 2^p(2^x - 1).$$

This is a proper divisor of n if and only if either $x = p$ or $x = 1$.■
So, every even perfect number $m = 2^{p-1}(2^p - 1)$ generates two distinct near-perfect numbers $n_1 = 2m$ and $n_2 = 2^p m$. Note that n_1 is $\mathcal{P}_{p+1}$- near-perfect, while n_2 is not $\mathcal{P}$-near-perfect.

Quite another type of near-perfect numbers generated by perfect numbers gives the following theorem.

Theorem 5. *Number n of the form $n = 2^{p-1}(2^p - 1)^2$, where p and $2^p - 1$ are prime, is near-perfect.*

Proof. We have

$$\sigma(n) - 2n = (2^p - 1)((2^p - 1)^2 + (2^p - 1) + 1) - 2^p(2^p - 1)^2 = 2^p - 1.$$

Since $2^p - 1$ is a proper divisor of n, then n is near-perfect (which generated by perfect number $2^{p-1}(2^p - 1)^2$).■

Sequence (1.3) shows that near-perfect numbers with odd redundant divisors occur very rarely. We conjecture that all near-perfect numbers with odd redundant divisors have form as in Theorem 5.

Conjecture 2. *If the redundant divisor for an even near-perfect is odd, then it is Mersenne prime.*

Note that, from Theorems 4-5 it follows that every perfect number is represented as difference of two near-perfect numbers. Indeed, for every perfect number m, numbers $n_2 = 2^p m$ and $n_3 = (2^p - 1)m$ are near-perfect, such that $n_2 - n_3 = m$.
Besides, the following conjecture seems plausible.

Conjecture 3. *If number l is not a power of 2, then it could be redundant divisor for at most one near-perfect number.*

Note that, if Conjectures 2-3 are true, then we obtain the following Euler-like theorem.

Theorem 6. *If Conjectures $2-3$ are true, then every even near-perfect number n with odd redundant divisor has form $n = 2^{p-1}(2^p - 1)^2$, where p and $2^p - 1$ are primes.*

Proof. From Conjecture 2 we conclude that, redundant divisor $d(n)$ has form $d(n) = 2^p - 1$ with prime p and $2^p - 1$. Now from Conjecture 3 and Theorem 5 we conclude that n has the required form. ■
Remark. In 2010 (see sequence A181595 [4]) the author conjectured that all near-perfect numbers are even. In particular, it is easy to show that odd square-free numbers are never near-perfect. However, at the beginning of 2012 Donovan Johnson [2] found an only up to $2 \cdot 10^{12}$ odd near-perfect number which is $173369889 = 3^4 \cdot 7^2 \cdot 11^2 \cdot 19^2$.

References

[1] R. K. Guy, *Unsolved Problems in Number Theory,* 2-nd edition, Springer-Verlag, 1994.
[2] D. Johnson, Private communication, 2012.
[3] W. Sierpiński, Sur les numbers pseudoparfaits, Mat. Vesnik **2** (1965), 212-213.
[4] N. J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences* (*http* : *//www.research.att.com/* $\sim$ *njas*)

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