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THE TENTH PERFECT NUMBER.

By R. E. POWERS, Denver, Colorado.

A number which is equal to the sum of all its divisors is called a "perfect number." Thus, the divisors of 6 are 1, 2, and 3, the sum of which is equal to 6; the divisors of 28 are 1, 2, 4, 7, and 14, whose sum is 28. Euclid (IX, 36) proved that if 2^p-1 is prime, then $2^{p-1}(2^p-1)$ is a perfect number, and no other perfect numbers are known. In 1644 Mersenne, in the preface to his *Cogitata Physico-Mathematica*, stated, in effect, that the only values of p not greater than 257 which make 2^p-1 prime are 2, 3, 5, 7, 13, 17, 19, 31, 67, 127, and 257. Regarding these "Mersenne's Numbers" (2^p-1), W. W. Rouse Ball, in his *Mathematical Recreations and Essays* (4th Edition, pages 262, 263, 269), says:

"I assume that the number 67 is a misprint for 61. With this correction, we have no reason to doubt the truth of the statement, but it has not been definitely established. . . . Seelhoff showed that 2^p-1 is prime when $p=61$, . . . and Cole gave the factors when $p=67$. . . . One of the unsolved riddles of higher arithmetic . . . is the discovery of the method by which Mersenne or his contemporaries determined values of p which make a number of the form 2^p-1 a prime. . . . The riddle is still, after nearly 250 years, unsolved."

No exception to Mersenne's assertion (corrected by the substitution of 61 for 67) is known at the present time. Below we show, however, that $2^{89}-1$ is a prime number, contrary to his statement.

Following is a list of Mersenne's Numbers thus far proved to be prime, with the corresponding perfect numbers:

p	2^p-1	*Perfect Numbers*
2	3	6
3	7	28
5	31	496
7	127	8,128
13	8,191	33,550,336
17	131,071	8,589,869,056
19	524,287	137,438,691,328
31	2,147,483,647	(19 digits)
61	2,305,843,009,213,693,951	(37 digits)

To these must now be added the prime number $2^{89}-1$, so that the tenth perfect number is $2^{88}(2^{89}-1)$, or

191561942608236107294793378084303638130997321548169216

(it is known that 2^p-1 is composite for all other values of p not greater than 100).

In his *Théorie des Nombres,* page 376, Lucas says: "Nous pensons avoir démontré par de très longs calculs qu'il n'existe pas de nombres parfaits pour $p=67$ et $p=89$." While this result has since been verified for $p=67$, the opinion has been expressed that also the case $p=89$ needed an independent examination. The result here shown that $2^{89}-1$ is a prime is therefore in conflict with Lucas' computation. The same writer, in an article entitled "Théorie des Fonctions Numériques Simplement Périodiques," Section XXIX, in the *American Journal of Mathematics,* Volume 1 (1878), proved the following remarkable theorem (the theorem appears on page 316 of the volume):

"If $P=2^{4q+1}-1$, and we form the series of residues (modulo P)

$$4, 14, 194, 37634, \ldots,$$

each of which is equal to the square of the preceding, diminished by two units: the number P is composite if none of the $4q+1$ first residues is equal to O; P is prime if the first residue O lies between the $2q$th and the $(4q+1)$th term."

Applying the above theorem to the number $2^{89}-1$, and denoting the terms of the series by L_1, L_2, L_3, ..., we found the following residues (modulo $2^{89}-1$):

m	L_m
1	4
2	14
3	194
.	. . . .
10	—115,113,975,804,653,882,052,836,464
20	36,000,517,785,442,762,303,479,300
30	—204,144,540,641,167,292,618,604,303
40	−126,791,709,316,676,382,795,042,761
50	−90,990,560,635,837,660,454,542,648
60	−206,308,592,424,355,282,693,419,690
70	99,498,791,857,820,493,810,407,653
80	269,783,273,665,984,523,074,966,550
.	. . . .
86	−309,403,333,482,440,150,628,882,422
87	−35,184,372,088,832
88	0

Since the first (and only) residue 0 occurs at the 88th term of the above series, it follows, from the foregoing theorem, that $2^{89}-1$, or

618,970,019,642,690,137,449,562,111

is a PRIME NUMBER.

As M. Lucas points out, his method used above is free from any uncertainty as to the accuracy of the conclusion that the number under consideration is prime, in case our attempt to arrive at the residue 0 meets with success, since an error in calculating any term of the series would have the effect of preventing the appearance of the residue 0. We would add that, denoting the number $2^{89}-1$ by N, we have verified that

$$3^{N-1}-1 \text{ is divisible by } N,$$

which is in accordance with Fermat's well-known theorem.

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